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Research Article

Some classifications for Gauss map of Tubular hypersurfaces in $\mathbb{E}_1^4$ concerning linearized operators $\mathcal{L}_k$

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Abstract

In this study, we deal with the Gauss map of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space concerning the linearized operators $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$. We obtain the $\mathcal{L}_1$ (Cheng-Yau) operator of the Gauss map of tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on timelike or spacelike curves with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map, second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1, 2\}$.

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1. Introduction

Let (M, g) be a hypersurface of $(n+1)$-dimensional Minkowski space $\mathbb{E}_1^{n+1}$, Δ denote its Laplace operator. A smooth mapping $\phi : M \to \mathbb{E}_1^{n+1}$ is said to be *finite type* if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

where ϕ_0 is a constant vector and ϕ_i is an eigenvector of Δ corresponding to the eigenvector λ_i for $i = 1, 2, \ldots, k$. More precisely, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then ψ is said to be *k-type* ([4, 6, 7]). Several results on the study of finite type mappings were summed up in a report by B.-Y. Chen in [5] (See also [8, 24]).

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Let N denote the Gauss map of M. From the definition above, one can conclude that N is of *1-type* if and only if it satisfies the equation

$$\Delta N = \lambda(N + C) \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a constant vector C. However, Gauss map of some important submanifolds such as catenoid and helicoid of the Euclidean 3-space $\mathbb{E}^3$ satisfies

$$\Delta N = f(N + C) \tag{1.2}$$

which is weaker than (1.1), where $f \in C^\infty(M)$ is a smooth function, [10]. These submanifolds whose Gauss map N satisfying (1.2) are said to have *pointwise 1-type Gauss map*. Submanifolds with pointwise 1-type Gauss map have been worked in several papers (cf. [10, 20, 23, 24]).

On the other hand, the Gauss map of some hypersurfaces of semi-Euclidean spaces satisfies the equation

$$\Delta N = f_1 N + f_2 C \tag{1.3}$$

for some smooth functions f_1, f_2 and a constant vector C. A submanifold is said to have *generalized 1-type Gauss map* if its Gauss map satisfies the condition (1.3), [25]. After this definition was given, hypersurfaces of pseudo-Euclidean spaces have been considered in terms of having generalized 1-type Gauss map, [17, 19, 25, 26].

In the recent years, the definition of $\mathcal{L}_k$-finite type maps has been obtained by replacing Δ in the definition above with the sequence of operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_{n-1}$, [1,2]. Note that, by the definition of these operators, one can obtain $\mathcal{L}_0 = -\Delta$ and $\mathcal{L}_1 = \Box$ is called as the Cheng-Yau operator introduced in [9]. By motivating this idea, notion of $\mathcal{L}_k$-pointwise 1-type Gauss map and generalized $\mathcal{L}_k$ 1-type Gauss map was presented in [14] and [18], respectively (see Definition 2.1). After the case $k = 1$ is studied in these papers, many result obtained on hypersurfaces with certain type of Gauss map, [11–13, 19, 22, 25, 26].

On the other hand, in [3], the general expression of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in $\mathbb{E}_1^4$ has been given and their some geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures have been obtained. Also, tubular hypersurfaces in $\mathbb{E}_1^4$ by taking constant radius function have been studied in [3].

In this paper, we study the tubular hypersurfaces in Lorentz-Minkowski 4-space $\mathbb{E}_1^4$ with the aid of $\mathcal{L}_k$ operators, $k \in \{1, 2\}$. In Sect. 2, we give basic notation, facts and definitions about hypersurfaces of Minkowski spaces. In Sect. 3 and Sect 4, we consider some classifications of tubular hypersurfaces by considering their Gauss maps in terms of their types with respect to the operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

2. Preliminaries

Let $\mathbb{E}_1^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space with the canonical pseudo-Euclidean metric $\langle\ ,\rangle$ of index 1 and signature $(-,+,+,...,+)$ given by

$$\langle\ ,\rangle = -dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_{n+1}^2$$

where $(x_1, x_2, ..., x_{n+1})$ is a rectangular coordinate system in $\mathbb{E}_1^{n+1}$.

If $\Gamma : M \longrightarrow \mathbb{E}_1^{n+1}$ is an isometric immersion from an n-dimensional orientable manifold M to $\mathbb{E}_1^{n+1}$, then the induced metric on M by the immersion Γ can be Riemannian or Lorentzian. Let N denotes a unit normal vector field and put $\langle N, N\rangle = \varepsilon = \pm 1$, so that $\varepsilon = 1$ or $\varepsilon = -1$ according to M is endowed with a Lorentzian or Riemannian metric, respectively.

The operator $\mathcal{L}_k$ acting on the coordinate functions of the Gauss map N of the hypersurface M in $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$ is

$$\mathcal{L}_k N = -\varepsilon \mathfrak{C}_k \left(\nabla H_{k+1} + \left(n H_1 H_{k+1} - (n-k-1) H_{k+2} \right) N \right). \tag{2.1}$$

Here,

$$\binom{n}{k} H_k = (-\varepsilon)^k a_k, \quad \left(\binom{n}{k} = \frac{n!}{k!(n-k)!} \right), \tag{2.2}$$

such that

$$\left.\begin{array}{l} a_1 = -\sum_{i=1}^{n} \kappa_i, \\ a_k = (-1)^k \sum_{i_1<i_2<\ldots<i_k}^{n} \kappa_{i_1}\kappa_{i_2}\ldots\kappa_{i_k}, \ k = 2,3,\ldots,n \end{array}\right\} \tag{2.3}$$

and H_k is called the k-th mean curvature of order k of M.

Also, the constant $\mathfrak{C}_k$ is given by

$$\mathfrak{C}_k = \binom{n}{k+1} (-\varepsilon)^k. \tag{2.4}$$

(For more details about the linearized operator $\mathcal{L}_k$, one can see [16].)

Definition 2.1. Let $\mathfrak{m}$ and $\mathfrak{n}$ be non-zero smooth functions on M, $C \in \mathbb{E}_1^{n+1}$ be a non-zero constant vector and $k \in \{0,1,2,\ldots,n\}$.

If the Gauss map N of an oriented submanifold M in $\mathbb{E}_1^4$ satisfies

i: $\mathcal{L}_k N = \mathfrak{m} N + \mathfrak{n} C$, then M has generalized $\mathcal{L}_k$ 1-type Gauss map;
ii: $\mathcal{L}_k N = \mathfrak{m} N$, then M has first kind $\mathcal{L}_k$-pointwise 1-type Gauss map;
iii: $\mathcal{L}_k N = \mathfrak{m}(N + C)$, then M has second kind $\mathcal{L}_k$-pointwise 1-type Gauss map;
iv: $\mathcal{L}_k N = 0$, then N is called $\mathcal{L}_k$-harmonic.

In this study, we will deal with Gauss maps of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space $\mathbb{E}_1^4$ concerning linearized operators $\mathcal{L}_1$ and $\mathcal{L}_2$. So, let us give some notions in $\mathbb{E}_1^4$.

Let $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ be three vectors in $\mathbb{E}_1^4$. The inner product and vector product are defined by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \tag{2.5}$$

and

$$\overrightarrow{u} \times \overrightarrow{v} \times \overrightarrow{w} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{2.6}$$

respectively. Here $e_i, (i = 1,2,3,4)$ are standard basis vectors.

A vector $\overrightarrow{u} \in \mathbb{E}_1^4$ is called spacelike, timelike or lightlike (null) if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle > 0$ (or $\overrightarrow{u} = 0$), $\langle \overrightarrow{u}, \overrightarrow{u} \rangle < 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 0$, respectively. A curve $\beta(s)$ in $\mathbb{E}_1^4$ is spacelike, timelike or lightlike (null), if all its velocity vectors $\beta'(s)$ are spacelike, timelike or lightlike, respectively and a non-null (i.e. timelike or spacelike) curve has unit speed if $\langle \beta', \beta' \rangle = \mp 1$. Also, the norm of the vector $\overrightarrow{u}$ is $\| \overrightarrow{u} \| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle|}$ [15].

Let F_1, F_2, F_3, F_4 be unit tangent vector field, principal normal vector field, binormal vector field, trinormal vector field of a timelike or spacelike curve $\beta(s)$, respectively and $\{F_1, F_2, F_3, F_4\}$ be the moving Frenet frame along $\beta(s)$ in $\mathbb{E}_1^4$. The Frenet equations can be given according to the causal characters of non-null Frenet vector fields F_1, F_2, F_3 and F_4 as follows [21]:

If the curve $\beta(s)$ is timelike, i.e. $\langle F_1, F_1\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 2, 3, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = k_1 F_1 + k_2 F_3, \\ F_3' = -k_2 F_2 + k_3 F_4, \\ F_4' = -k_3 F_3; \end{array}\right\} \tag{2.7}$$

if the curve $\beta(s)$ is spacelike with timelike principal normal vector field F_2, i.e. $\langle F_2, F_2\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 3, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = k_1 F_1 + k_2 F_3, \\ F_3' = k_2 F_2 + k_3 F_4, \\ F_4' = -k_3 F_3; \end{array}\right\} \tag{2.8}$$

if the curve $\beta(s)$ is spacelike with timelike binormal vector field F_3, i.e. $\langle F_3, F_3\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3; \end{array}\right\} \tag{2.9}$$

if the curve $\beta(s)$ is spacelike with timelike trinormal vector field F_4, i.e. $\langle F_4, F_4\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 3)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = -k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3. \end{array}\right\} \tag{2.10}$$

Here k_1, k_2, k_3 are the first, second and third curvatures of the non-null curve $\beta(s)$.

Also, if p is a fixed point in $\mathbb{E}_1^4$ and r is a positive constant, then the pseudo-Riemannian hypersphere and the pseudo-Riemannian hyperbolic space are defined by

$$S_1^3(p, r) = \{x \in \mathbb{E}_1^4 : \langle x - p, x - p\rangle = r^2\}$$

and

$$H_0^3(p, r) = \{x \in \mathbb{E}_1^4 : \langle x - p, x - p\rangle = -r^2\},$$

respectively.

If M is an oriented hypersurface in $\mathbb{E}_1^4$, then the gradient of a smooth function $f(s, t, w)$, defined on M, can be obtained by

$$\nabla f = \frac{1}{\mathfrak{g}} \begin{pmatrix} \left(\left(g_{23}^2 - g_{22}g_{33}\right) f_s + \left(-g_{13}g_{23} + g_{12}g_{33}\right) f_t + \left(g_{13}g_{22} - g_{12}g_{23}\right) f_w\right) \partial s \\ + \left(\left(-g_{13}g_{23} + g_{12}g_{33}\right) f_s + \left(g_{13}^2 - g_{11}g_{33}\right) f_t + \left(-g_{12}g_{13} + g_{11}g_{23}\right) f_w\right) \partial t \\ + \left(\left(g_{13}g_{22} - g_{12}g_{23}\right) f_s + \left(-g_{12}g_{13} + g_{11}g_{23}\right) f_t + \left(g_{12}^2 - g_{11}g_{22}\right) f_w\right) \partial w \end{pmatrix}, \tag{2.11}$$

where

$$\mathfrak{g} = g_{13}^2 g_{22} - 2g_{12}g_{13}g_{23} + g_{11}g_{23}^2 + g_{12}^2 g_{33} - g_{11}g_{22}g_{33};$$

$\{s, t, w\}$ is a local coordinat system of M; f_s, f_t, f_w are the partial derivatives of f and $g_{11} = \langle \partial s, \partial s\rangle$, $g_{12} = \langle \partial s, \partial t\rangle$, $g_{13} = \langle \partial s, \partial w\rangle$, $g_{22} = \langle \partial t, \partial t\rangle$, $g_{23} = \langle \partial t, \partial w\rangle$, $g_{33} = \langle \partial w, \partial w\rangle$.

3. Some classifications for Tubular hypersurfaces generated by timelike curves with L_k operators in $\mathbb{E}_1^4$

In this section, we obtain the $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss map of the tubular hypersurfaces $\mathcal{T}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1,2\}$.

The tubular hypersurfaces $\mathcal{T}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ can be parametrized by

$$\mathcal{T}(s,t,w) = \beta(s) + r(\cos t \cos w F_2(s) + \sin t \cos w F_3(s) + \sin w F_4(s)). \tag{3.1}$$

The unit normal vector field of (3.1) is

$$N = -(\cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4) \tag{3.2}$$

and so,

$$\langle N, N\rangle = 1. \tag{3.3}$$

The coefficients of the first fundamental form of (3.1) are

$$\left.\begin{array}{l} g_{11} = (rk_2 \cos t \cos w - rk_3 \sin w)^2 + r^2\left(k_2^2+k_3^2\right)\sin^2 t \cos^2 w - (1 + rk_1 \cos t \cos w)^2, \\ g_{12} = g_{21} = r^2(k_2 \cos w - k_3 \cos t \sin w)\cos w,\ g_{22} = r^2 \cos^2 w, \\ g_{13} = g_{31} = r^2 k_3 \sin t,\ g_{23} = g_{32} = 0,\ g_{33} = r^2. \end{array}\right\} \tag{3.4}$$

The principal curvatures of (3.1) are

$$\kappa_1 = \kappa_2 = \frac{1}{r},\ \kappa_3 = \frac{k_1 \cos t \cos w}{1 + rk_1 \cos t \cos w}. \tag{3.5}$$

For more details about these hypersurfaces, one can see [3].

3.1. Some classifications for Tubular hypersurfaces generated by timelike curves with $\mathcal{L}_1$ (Cheng-Yau) operator in $\mathbb{E}_1^4$

The functions a_k of the tubular hypersurfaces (3.1) in $\mathbb{E}_1^4$ are obtained from (2.3) and (3.5) by

$$a_1 = \frac{-2 - 3rk_1 \cos t \cos w}{r(1 + rk_1 \cos t \cos w)},\ a_2 = \frac{1 + 3rk_1 \cos t \cos w}{r^2(1 + rk_1 \cos t \cos w)},\ a_3 = -\frac{k_1}{r^2(rk_1 + \sec t \sec w)}. \tag{3.6}$$

Also, from (2.11), (3.4) and (3.6), we have

$$\begin{aligned} \nabla a_2 = &-\frac{2\left(k_1' \cos t + k_1 k_2 \sin t\right)\cos w}{r(1 + rk_1 \cos t \cos w)^3} F_1 - \frac{k_1\left(2\cos^2 t \cos(2w) + \cos(2t) - 3\right)}{2r^2(1 + rk_1 \cos t \cos w)^2} F_2 \\ &- \frac{2k_1 \sin t \cos t \cos^2 w}{r^2(1 + rk_1 \cos t \cos w)^2} F_3 - \frac{2k_1 \cos t \sin w \cos w}{r^2(1 + rk_1 \cos t \cos w)^2} F_4. \end{aligned} \tag{3.7}$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.7), we reach that

$$\begin{aligned} \mathcal{L}_1 N = &-\frac{2\left(k_1 k_2 \sin t + k_1' \cos t\right)\cos w}{r(1 + rk_1 \cos t \cos w)^3} F_1 \\ &- \frac{2\left(rk_1\left(3rk_1 \cos^3 t \cos^3 w + 2\cos^2 t \cos(2w) + \cos(2t)\right) + \cos t \cos w\right)}{r^3(1 + rk_1 \cos t \cos w)^2} F_2 \\ &- \frac{2(1 + 3rk_1 \cos t \cos w)\sin t \cos w}{r^3(1 + rk_1 \cos t \cos w)} F_3 - \frac{2(3rk_1 \cos t \cos w + 1)\sin w}{r^3(1 + rk_1 \cos t \cos w)} F_4. \end{aligned} \tag{3.8}$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_1$ 1-type Gauss map, first kind $\mathcal{L}_1$-pointwise 1-type Gauss map, second kind $\mathcal{L}_1$-pointwise 1-type Gauss map and $\mathcal{L}_1$-harmonic Gauss map.

Let the tubular hypersurfaces $\mathfrak{T}(s,t,w)$ have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}N + \mathfrak{n}C$, where $C = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.7) we obtain that

$$\left.\begin{array}{l} C_1' + C_2k_1 = 0, \\ C_2' + C_1k_1 - C_3k_2 = 0, \\ C_3' + C_2k_2 - C_4k_3 = 0, \\ C_4' + C_3k_3 = 0. \end{array}\right\} \tag{3.9}$$

Also, by taking derivatives the constant vector C with respect to t and w separately, one can see that the functions C_i depend only on s.

Firstly, let us classify the tubular hypersurfaces $\mathfrak{T}(s,t,w)$ which have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map.

From (3.2) and (3.8), we get

$$\left.\begin{array}{l} -\frac{2\left(k_1' \cos t + k_1k_2 \sin t\right)\cos w}{r(1+rk_1 \cos t \cos w)^3} = \mathfrak{n}C_1, \\ -\frac{2\left(rk_1\left(3rk_1 \cos^3 t \cos^3 w + 2\cos^2 t \cos(2w) + \cos(2t)\right) + \cos t \cos w\right)}{r^3(1+rk_1 \cos t \cos w)^2} = \mathfrak{m}\left(-\cos t \cos w\right) + \mathfrak{n}C_2, \\ -\frac{2(1+3rk_1 \cos t \cos w)\sin t \cos w}{r^3(1+rk_1 \cos t \cos w)} = \mathfrak{m}\left(-\sin t \cos w\right) + \mathfrak{n}C_3, \\ -\frac{2(1+3rk_1 \cos t \cos w)\sin w}{r^3(1+rk_1 \cos t \cos w)} = \mathfrak{m}\left(-\sin w\right) + \mathfrak{n}C_4. \end{array}\right\} \tag{3.10}$$

Now, let us investigate the non-zero functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ from the above four equations.

Firstly, let us assume that $C_1 \neq 0$.

In this case, from the first equation of (3.10) it's easy to see that

$$\mathfrak{n}(s,t,w) = -\frac{2\left(k_1' \cos t + k_1k_2 \sin t\right)\cos w}{r(1+rk_1 \cos t \cos w)^3 C_1}. \tag{3.11}$$

Here, when the equation (3.11) is successively substituted into the second, third and fourth equations of (3.10), we obtain

$$\mathfrak{m}(s,t,w) = \frac{2\left(\begin{array}{l} \left(rk_1\left(\begin{array}{l} 3rk_1 \cos^3 t \cos^3 w \\ +2\cos^2 t \cos(2w) + \cos(2t) \end{array}\right) + \cos t \cos w\right) C_1(1+rk_1 \cos t \cos w) \\ -C_2r^2\left(k_1' \cos t + k_1k_2 \sin t\right)\cos w \end{array}\right)}{C_1r^3(1+rk_1 \cos t \cos w)^3 \cos t \cos w},$$

$$\mathfrak{m}(s,t,w) = \frac{2\left(C_1(1+rk_1 \cos t \cos w)^2(1+3rk_1 \cos t \cos w) - C_3r^2\left(k_1' \cot t + k_1k_2\right)\right)}{C_1r^3(1+rk_1 \cos t \cos w)^3},$$

$$\mathfrak{m}(s,t,w) = \frac{2\left(C_1(1+rk_1 \cos t \cos w)^2(1+3rk_1 \cos t \cos w) - C_4r^2\left(k_1' \cos t + k_1k_2 \sin t\right)\cot w\right)}{C_1r^3(1+rk_1 \cos t \cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$(C_3 - C_4 \sin t \cot w)\left(k_1' \cot t + k_1 k_2\right) = 0, \tag{3.12}$$

$$k_1(C_1 \sec t \sec w + k_2 r(C_2 \tan t - C_4 \sin t \cot w)) + r(C_1 k_1^2 - k_1'(C_4 \cos t \cot w - C_2)) = 0, \tag{3.13}$$

$$k_1(rk_2(C_3 - C_2 \tan t) - C_1 \sec t \sec w) - C_1 r k_1^2 + r k_1'(C_3 \cot t - C_2) = 0. \tag{3.14}$$

In the equation (3.12), it holds that $k_1' \cot t + k_1 k_2 \neq 0$. This is because, when $k_1' \cot t + k_1 k_2 = 0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.10) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_1 N = \mathfrak{m} N + \mathfrak{n} C$. So, from the equation (3.12) and $k_1' \cot t + k_1 k_2 \neq 0$, we have $C_3 = C_4 = 0$. When $C_3 = C_4 = 0$, substituting this into the equation (3.14) yields

$$\left(C_1 r k_1^2 + C_2 r k_1'\right) \cos t + C_1 k_1 \sec w + C_2 r k_1 k_2 \sin t = 0.$$

Thus, we have

$$C_1 k_1^2 + C_2 k_1' = C_1 k_1 = C_2 k_1 k_2 = 0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.15}$$

If $k_1 = 0$ in (3.15), then from the second, third and fourth equations of (3.10), it is calculated as

$$\left.\begin{aligned} C_2 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \cos t \cos w, \\ C_3 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \sin t \cos w, \\ C_4 r^3 \mathfrak{n}(s,t,w) &= \left(\mathfrak{m}(s,t,w) r^3 - 2\right) \sin w, \end{aligned}\right\} \tag{3.16}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{n}(s,t,w)$ in (3.16).

Now, let us assume that $C_2 = 0$ in (3.15). In this case, from the second equation of (3.10), it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{2\left(\cos t \cos w + r k_1 \left(\cos(2t) + 3 r k_1 \cos^3 t \cos^3 w + 2 \cos^2 t \cos(2w)\right)\right)}{r^3 (1 + r k_1 \cos t \cos w)^2 \cos t \cos w}. \tag{3.17}$$

Here, when the equation (3.17) is successively substituted into the third and fourth equations of (3.10), we obtain

$$\mathfrak{n}(s,t,w) C_3 = \frac{-2k_1}{r^2 (1 + r \cos t \cos w k_1)^2} \tan t,$$

$$\mathfrak{n}(s,t,w) C_4 = \frac{-2k_1}{r^2 (1 + r k_1 \cos t \cos w)^2} \sec t \tan w.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Hence, we can state the following theorem:

Theorem 3.1. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_1$ 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}\,(N + C)$.

From (3.2) and (3.8), we get

$$\left.\begin{aligned}
&-\tfrac{2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}=\mathfrak{m}C_1,\\
&-\tfrac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w+2\cos^2 t\cos(2w)+\cos(2t)\right)+\cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2}=\mathfrak{m}\left(-\cos t\cos w+C_2\right),\\
&-\tfrac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin t\cos w+C_3\right),\\
&-\tfrac{2(1+3rk_1\cos t\cos w)\sin w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin w+C_4\right).
\end{aligned}\right\}\tag{3.18}$$

Here, from the fourth equation of (3.18) it's easy to see that

$$\mathfrak{m}(s,t,w)=-\frac{2\left(1+3rk_1\cos t\cos w\right)\sin w}{r^3(1+rk_1\cos t\cos w)\left(-\sin w+C_4\right)}.\tag{3.19}$$

When the equation (3.19) is successively substituted into the second and third equations of (3.18), we obtain

$$\begin{aligned}
&3C_2r^2k_1^2\cos^2 t\cos^2 w+4C_2rk_1\cos t\cos w+C_2-rk_1=0,\\
&C_4\sin t\cos w-C_3\sin w=0.
\end{aligned}$$

So, we have

$$k_1=C_2=C_3=C_4=0.\tag{3.20}$$

Now, when the components of the equation (3.20) is substituted into the second, third or fourth equations of (3.18), we calculated

$$\mathfrak{m}(s,t,w)=\frac{2}{r^3}.\tag{3.21}$$

Also, from the first equation of (3.18) and (3.21), we have $C_1=0$.

From the calculations made above for classify the tubular hypersurfaces $\mathfrak{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1N=\mathfrak{m}\left(N+C\right)$, we can give the following theorem:

Theorem 3.2. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with second kind $\mathcal{L}_1$-pointwise 1-type Gauss map.*

Moreover, if the function m is constant in Definition 2.1 (**ii** or **iii**), then we say M has first or second kind $\mathcal{L}_k$-(global) pointwise 1-type Gauss map. Thus, we can state the following theorem:

Theorem 3.3. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, have first kind $\mathcal{L}_1$-(global) pointwise 1-type Gauss map, i.e., $\mathcal{L}_1N=\mathfrak{m}N$ if and only if $k_1=0$, where $\mathfrak{m}(s,t,w)=\frac{2}{r^3}$.*

Finally, in the equation (3.8), since all of the coefficients of F_1, F_2, F_3 and F_4 cannot be zero, we can give the following theorem:

Theorem 3.4. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, cannot have $\mathcal{L}_1$-harmonic Gauss map.*

3.2. Some classifications for Tubular hypersurfaces generated by timelike curves with $\mathcal{L}_2$ operator in $\mathbb{E}_1^4$

Firstly, it is calculated from (2.11), (3.4) and (3.6) as

$$\begin{aligned}\nabla a_3 =&\frac{(k_1'\cos t + k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1 + \frac{k_1\left(2\cos^2 t\cos(2w)+\cos(2t)-3\right)}{4r^3(1+rk_1\cos t\cos w)^2}F_2\\ &+\frac{k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)^2}F_3+\frac{k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)^2}F_4.\end{aligned}\tag{3.22}$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.22), we have

$$\begin{aligned}\mathcal{L}_2N =& \frac{(k_1'\cos t + k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1\\ &+\frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w+12\cos^3 t\cos(3w)\\ +19\cos t\cos w+9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3}F_2\\ &+\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)}F_3+\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)}F_4.\end{aligned}\tag{3.23}$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_2$ 1-type Gauss map, first kind $\mathcal{L}_2$-pointwise 1-type Gauss map, second kind $\mathcal{L}_2$-pointwise 1-type Gauss map and $\mathcal{L}_2$-harmonic Gauss map.

Now, let us classify the tubular hypersurfaces $\mathfrak{T}(s,t,w)$ which have generalized $\mathcal{L}_2$ 1-type Gauss map. From (3.2) and (3.23), we get

$$\left.\begin{array}{l}\frac{\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r^2(1+rk_1\cos t\cos w)^3}=\mathfrak{n}C_1,\\ \\ \frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w+12\cos^3 t\cos(3w)\\ +19\cos t\cos w+9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3}=\mathfrak{m}\left(-\cos t\cos w\right)+\mathfrak{n}C_2,\\ \\ \frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin t\cos w\right)+\mathfrak{n}C_3,\\ \\ \frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin w\right)+\mathfrak{n}C_4.\end{array}\right\}\tag{3.24}$$

Firstly, let us assume that $C_1\neq 0$.

In this case, from the first equation of (3.24) it's easy to see that

$$\mathfrak{n}(s,t,w)=\frac{(k_1'\cos t+k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3C_1}.\tag{3.25}$$

Here, when the equation (3.25) is successively substituted into the second, third and fourth equations of (3.24), we obtain

$$\mathfrak{m}(s,t,w)=\frac{\left(\begin{array}{l}2k_1(-3C_1\cos t\cos w+C_1\sec t\sec w+C_2rk_2\tan t)+2C_2rk_1'\\ -6C_1r^2k_1^3\cos^3 t\cos^3 w-C_1rk_1^2\left(6\cos^2 t\cos(2w)+3\cos(2t)+1\right)\end{array}\right)}{2C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w)=\frac{C_3r\left(k_1'\cot t+k_1k_2\right)-3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w)=\frac{C_4r\left(k_1'\cos t+k_1k_2\sin t\right)\cot w-3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$(C_4 \sin t \cos w - C_3 \sin w)\,(k_1' \cos t + k_1 k_2 \sin t) = 0, \tag{3.26}$$

$$k_1(C_1 \sec t \sec w + r k_2(C_2 \tan t - C_4 \sin t \cot w)) + C_1 r k_1^2 + r k_1'(C_2 - C_4 \cos t \cot w) = 0, \tag{3.27}$$

$$k_1(C_1 \sec t \sec w + r k_2(C_2 \tan t - C_3)) + C_1 r k_1^2 + r k_1'(C_2 - C_3 \cot t) = 0. \tag{3.28}$$

In the equation (3.26), it holds that $k_1' \cos t + k_1 k_2 \sin t \neq 0$. This is because when $k_1' \cos t + k_1 k_2 \sin t = 0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.24) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_2 N = \mathfrak{m} N + \mathfrak{n} C$. So, from the equation (3.26) and $k_1' \cos t + k_1 k_2 \sin t \neq 0$, we have $C_3 = C_4 = 0$. When $C_3 = C_4 = 0$, substituting this into the equation (3.28) yields

$$r\left(C_1 k_1^2 + C_2 k_1'\right) \cos t + C_2 r k_1 k_2 \sin t + C_1 k_1 \sec w = 0.$$

Thus, we have

$$C_1 k_1^2 + C_2 k_1' = C_1 k_1 = C_2 k_1 k_2 = 0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.29}$$

If $k_1 = 0$ in (3.29), then from the second, third and fourth equations of (3.24), it is calculated as

$$\left.\begin{array}{l} \mathfrak{m}(s,t,w)\cos t \cos w = \mathfrak{n}(s,t,w) C_2, \\ \mathfrak{m}(s,t,w)\sin t \cos w = \mathfrak{n}(s,t,w) C_3, \\ \mathfrak{m}(s,t,w)\sin w = \mathfrak{n}(s,t,w) C_4, \end{array}\right\} \tag{3.30}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ in (3.30).

Now, let us assume that $C_2 = 0$ in (3.29). In this case, from the second equation of (3.24), it's easy to see that

$$\mathfrak{m}(s,t,w) = -\frac{k_1 \left(\begin{array}{l} r k_1 \left(\begin{array}{l} 24 r k_1 \cos^4 t \cos^4 w + 12 \cos^3 t \cos(3w) \\ +19 \cos t \cos w + 9 \cos(3t) \cos w \end{array} \right) \\ +12 \cos^2 t \cos(2w) + 6 \cos(2t) - 2 \end{array} \right)}{8 r^3 (1 + r k_1 \cos t \cos w)^3 \cos t \cos w}. \tag{3.31}$$

Here, when the equation (3.31) is successively substituted into the third and fourth equations of (3.24), we obtain

$$\mathfrak{n}(s,t,w) C_3 = \tfrac{k_1(r k_1 \sin t \cos w + \tan t)}{r^3 (1 + r k_1 \cos t \cos w)^3},$$

$$\mathfrak{n}(s,t,w) C_4 = \tfrac{k_1(r k_1 \sin w + \sec t \tan w)}{r^3 (1 + r k_1 \cos t \cos w)^3}.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Therefore, we can give the following theorem:

Theorem 3.5. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_2$ 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathfrak{T}(s,t,w)$ which have second kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m}\,(N + C)$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l} \frac{\left(k_1' \cos t + k_1 k_2 \sin t\right) \cos w}{r^2(1+rk_1 \cos t \cos w)^3} = \mathfrak{m} C_1, \\ k_1 \dfrac{\left(\frac{rk_1}{2} \left(\begin{array}{l} 24rk_1 \cos^4 t \cos^4 w + 12 \cos^3 t \cos(3w) \\ +19 \cos t \cos w + 9 \cos(3t) \cos w \end{array} \right) \right)}{\left(+6 \cos^2 t \cos(2w) + 3 \cos(2t) - 1 \right)} = \mathfrak{m} \left(-\cos t \cos w + C_2 \right), \\ \frac{4r^3(1+rk_1 \cos t \cos w)^3}{3k_1 \sin t \cos t \cos^2 w} = \mathfrak{m} \left(-\sin t \cos w + C_3 \right), \\ \frac{r^3(1+rk_1 \cos t \cos w)}{3k_1 \cos t \sin w \cos w} = \mathfrak{m} \left(-\sin w + C_4 \right). \\ r^3(1+rk_1 \cos t \cos w) \end{array}\right\} \tag{3.32}$$

Here, from the last equation of (3.32) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{3k_1 \cos t \sin w \cos w}{r^3(1+rk_1 \cos t \cos w)\left(-\sin w + C_4\right)}. \tag{3.33}$$

Here, when the equation (3.33) is substituted into the second equation of (3.32), we obtain

$$-1 + 3C_2 \cos t \cos w + 3C_2 rk_1 \cos^2 t \cos^2 w = 0.$$

Since this is not possible, we can give the following theorem:

Theorem 3.6. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Now, let us classify the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have first kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m} N$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l} \frac{\cos w \left(\cos t k_1' + k_1 k_2 \sin t\right)}{r^2(1+rk_1 \cos t \cos w)^3} = 0, \\ k_1 \dfrac{\left(\frac{1}{2} rk_1 \left(\begin{array}{l} 24rk_1 \cos^4 t \cos^4 w + 12 \cos^3 t \cos(3w) \\ +19 \cos t \cos w + 9 \cos(3t) \cos w \end{array} \right) \right)}{\left(+6 \cos^2 t \cos(2w) + 3 \cos(2t) - 1 \right)} = \mathfrak{m} \left(-\cos t \cos w \right), \\ \frac{4r^3(1+rk_1 \cos t \cos w)^3}{3k_1 \sin t \cos t \cos^2 w} = \mathfrak{m} \left(-\sin t \cos w \right), \\ \frac{r^3(1+rk_1 \cos t \cos w)}{3k_1 \cos t \sin w \cos w} = \mathfrak{m} \left(-\sin w \right). \\ r^3(1+rk_1 \cos t \cos w) \end{array}\right\} \tag{3.34}$$

Here, from the last equation of (3.34) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{-3k_1 \cos t \cos w}{r^3(1+rk_1 \cos t \cos w)}. \tag{3.35}$$

Here, when the equation (3.35) is substituted into the second equation of (3.32), we obtain

$$k_1(rk_1 + \sec t \sec w) = 0.$$

Since this is not possible, we can give the following theorem:

Theorem 3.7. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Finally, since the coefficients F_1, F_2, F_3 and F_4 in equation (3.23) are all zero only for $k_1 = 0$, we can give the following theorem:

Theorem 3.8. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

4. Some classifications for Tubular hypersurfaces generated by spacelike curves with $\mathcal{L}_k$ operators in $\mathbb{E}_1^4$

In this section, we give the general formulas for $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss maps of the six types of tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on spacelike curves $\beta(s)$ with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1,2\}$.

The tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vectors F_i in $\mathbb{E}_1^4$ can be parametrized by

$$\left.\begin{array}{l}
\mathcal{T}^{\{2,1\}}(s,t,w) = \beta(s) + r(\cosh t \sinh w F_2(s) + \cosh w F_3(s) + \sinh t \sinh w F_4(s)),\\
\mathcal{T}^{\{2,-1\}}(s,t,w) = \beta(s) + r(\cosh t \cosh w F_2(s) + \sinh w F_3(s) + \sinh t \cosh w F_4(s)),\\
\mathcal{T}^{\{3,1\}}(s,t,w) = \beta(s) + r(\sinh t \sinh w F_2(s) + \cosh t \sinh w F_3(s) + \cosh w F_4(s)),\\
\mathcal{T}^{\{3,-1\}}(s,t,w) = \beta(s) + r(\sinh t \cosh w F_2(s) + \cosh t \cosh w F_3(s) + \sinh w F_4(s)),\\
\mathcal{T}^{\{4,1\}}(s,t,w) = \beta(s) + r(\cosh w F_2(s) + \sinh t \sinh w F_3(s) + \cosh t \sinh w F_4(s)),\\
\mathcal{T}^{\{4,-1\}}(s,t,w) = \beta(s) + r(\sinh w F_2(s) + \sinh t \cosh w F_3(s) + \cosh t \cosh w F_4(s)),
\end{array}\right\} \tag{4.1}$$

respectively. Here, we suppose for $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that

i) $\langle F_j, F_j\rangle = -1 = \varepsilon_j$ and for $i \neq j$, $\langle F_i, F_i\rangle = 1 = \varepsilon_i$, $i,j \in \{1,2,3,4\}$,

ii) if the tubular hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ *or* $\lambda = -1$, respectively (for more details, one can see [3]).

Now, let us write the following lemma which states the general parametric expressions of 6 different types of tubular hypersurfaces given by (4.1) and obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields in $\mathbb{E}_1^4$.

Lemma 4.1. *The general expression of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve $\beta(s)$ with non-null Frenet vectors $F_i(s)$ in $\mathbb{E}_1^4$ can be given by*

$$\mathcal{T}^{\{j,\lambda\}}(s,t,w) = \beta(s) + r\left(\sum_{i=2}^{4} \mu_i^{\lambda}(s,t,w) F_i(s)\right), \tag{4.2}$$

where

$$\mu_5^{\lambda}(s,t,w) = \mu_2^{\lambda}(s,t,w),\ \ \mu_6^{\lambda}(s,t,w) = \mu_3^{\lambda}(s,t,w)$$

and for $j = 2,3,4$

$$\mu_j^{\lambda}(s,t,w) = (\sinh w)^{\frac{1+\lambda}{2}} (\cosh w)^{\frac{1-\lambda}{2}} \cosh t,$$

$$\mu_{j+1}^{\lambda}(s,t,w) = (\sinh w)^{\frac{1-\lambda}{2}} (\cosh w)^{\frac{1+\lambda}{2}},$$

$$\mu_{j+2}^{\lambda}(s,t,w) = (\sinh w)^{\frac{1+\lambda}{2}} (\cosh w)^{\frac{1-\lambda}{2}} \sinh t.$$

Here, if the canal hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ or $\lambda = -1$, respectively.

Here, we can give the general parametric expressions of the unit normal vector fields, the coefficients of the first fundamental forms and the principal curvatures of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}$ parametrized by (4.2).

The unit normal vector fields $N^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$N^{\{j,\lambda\}} = -(-1)^{(4-j)!}\lambda^j \textstyle\sum_{i=2}^{4} \mu_i^\lambda F_i. \tag{4.3}$$

The coefficients of the first fundamental forms $g_{ik}^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$\left.\begin{aligned}
g_{11}^{\{j,\lambda\}} &= 1 + r^2(k_2)^2\left(-(-1)^{(4-j)!}\left(\mu_3^\lambda\right)^2 + (-1)^j\left(\mu_2^\lambda\right)^2\right)\\
&\quad + r^2(k_3)^2\left((-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2 + (-1)^j\left(\mu_4^\lambda\right)^2\right)\\
&\quad + 2(-1)^{(4-j)!}rk_1\mu_2^\lambda + r^2(k_1)^2\left(\mu_2^\lambda\right)^2 - 2(-1)^{(5-j)!}r^2k_2k_3\mu_2^\lambda\mu_4^\lambda,\\
g_{12}^{\{j,\lambda\}} &= g_{21}^{\{j,\lambda\}} = r^2(\mu_{j+1}^\lambda)_w\left((-1)^j k_3\left(\mu_2^\lambda\right)_w - (-1)^{(4-j)!}k_2\left(\mu_4^\lambda\right)_w\right),\\
g_{22}^{\{j,\lambda\}} &= r^2\left((\mu_{j+1}^\lambda)_w\right)^2,\\
g_{13}^{\{2,\lambda\}} &= g_{31}^{\{2,\lambda\}} = \lambda r^2\left(-k_2\cosh t + k_3\sinh t\right),\\
g_{13}^{\{3,\lambda\}} &= g_{31}^{\{3,\lambda\}} = -\lambda r^2 k_3\cosh t,\\
g_{13}^{\{4,\lambda\}} &= g_{31}^{\{4,\lambda\}} = \lambda r^2 k_2\sinh t,\\
g_{23}^{\{j,\lambda\}} &= g_{32}^{\{j,\lambda\}} = 0,\\
g_{33}^{\{j,\lambda\}} &= -\lambda r^2.
\end{aligned}\right\} \tag{4.4}$$

The principal curvatures $\kappa_i^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$\left.\begin{aligned}
\kappa_1^{\{j,\lambda\}} &= \kappa_2^{\{j,\lambda\}} = \tfrac{(-1)^{(4-j)!}\lambda^j}{r},\\
\kappa_3^{\{j,\lambda\}} &= \tfrac{k_1\mu_2^\lambda}{\lambda^j\left(1+(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}.
\end{aligned}\right\} \tag{4.5}$$

From Lemma 4.1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
\mathcal{L}_1 N^{\{j,\lambda\}} &= \frac{2\left(-(-1)^{(5-j)!}k_1k_2\mu_3^\lambda + k_1'\mu_2^\lambda\right)}{r\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^3}F_1\\
&\quad + \frac{-2\lambda\left(\mu_2^\lambda + 3r^2(\mu_2^\lambda)^3(k_1)^2 + k_1\left(\lambda r + 4(-1)^{(4-j)!}r(\mu_2^\lambda)^2\right)\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^2}F_2\\
&\quad + \frac{-2\lambda(-1)^{(4-j)!}\mu_3^\lambda\left(1 + 3(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)}F_3 + \frac{-2\lambda\mu_4^\lambda\left((-1)^{(4-j)!} + 3rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)}F_4.
\end{aligned} \tag{4.6}$$

Let $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ have generalized L_1 (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1 N^{\{j,\lambda\}} = \mathfrak{m}N^{\{j,\lambda\}} + \mathfrak{n}C$, where $C = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.8)-(2.10) we obtain for $\mathcal{T}^{\{j,\lambda\}}$ that

$$\left.\begin{aligned}
&C_1' + (-1)^{(4-j)!}C_2k_1 = 0,\\
&C_2' + C_1k_1 + (-1)^{(5-j)!}C_3k_2 = 0,\\
&C_3' + C_2k_2 - (-1)^{(4-j)!}C_4k_3 = 0,\\
&C_4' + C_3k_3 = 0.
\end{aligned}\right\} \tag{4.7}$$

Also, by taking derivatives the constant vector C with respect to t, w separately, one can see that the functions C_i depend only on s.

So, with similar procedure in Subsection 3.1, we obtain the following theorems:

Theorem 4.2. *There are no tubular hypersurfaces (4.2), obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_1$ 1-type Gauss map in $\mathbb{E}_1^4$.*

Theorem 4.3. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ with second kind $\mathcal{L}_1$-pointwise 1-type Gauss map in $\mathbb{E}_1^4$.*

Theorem 4.4. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ have first kind $\mathcal{L}_1$-(global) pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N^{\{j,\lambda\}} = \mathfrak{m} N^{\{j,\lambda\}}$ in $\mathbb{E}_1^4$ if and only if $k_1 = 0$, where $\mathfrak{m}(s,t,w) = \frac{2\lambda^{j+1}(-1)^{(4-j)!}}{r^3}$.*

Theorem 4.5. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ cannot have $\mathcal{L}_1$-harmonic Gauss map.*

Also, from Lemma 4.1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}\mathcal{L}_2 N^{\{j,\lambda\}} &= \frac{\lambda^j\left((-1)^j\mu_3^\lambda k_1 k_2 - (-1)^{(4-j)!}\mu_2^\lambda k_1'\right)}{r^2\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^3}F_1 \\ &+ \frac{-\lambda^{j+1}k_1\left(2\lambda(-1)^{(4-j)!} - 3(-1)^j\left(\mu_4^\lambda\right)^2 - 3(-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2 - 3(-1)^{(4-j)!}rk_1\left(\mu_2^\lambda\right)^3\right)}{r^3\left((-1)^{(4-j)!} + rk_1\mu_2^\lambda\right)^2}F_2 \\ &+ \frac{\lambda^{j+1}\mu_3^\lambda\left(3(-1)^{(5-j)!}rk_1\mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j rk_1\mu_2^\lambda\right)}F_3 + \frac{\lambda^{j+1}\mu_4^\lambda\left(3(-1)^{(5-j)!}rk_1\mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j rk_1\mu_2^\lambda\right)}F_4. \qquad (4.8)\end{aligned}$$

Thus, with similar procedure in Subsection 3.2, we can give the following theorems:

Theorem 4.6. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ with generalized $\mathcal{L}_2$ 1-type Gauss map in $\mathbb{E}_1^4$.*

Theorem 4.7. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}_1^4$.*

Theorem 4.8. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}_1^4$.*

Theorem 4.9. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}_1^4$ have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

Author Contributions

All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflicts Of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

No data was used for the research described in the article.

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